JOURNAL OF APPROXIMATION THEORY 56, 72–90 (1989)

Bernstein-Type Operators and Their Derivatives*

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Communicated by R. Bojanic

Received August 25, 1986

1. INTRODUCTION

The Bernstein-type integral operators discussed in this paper are given by

$$M_n f \equiv M_n(f, x) = (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_0^1 P_{n,k}(t) f(t)\, dt, \tag{1.1}$$

where $P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$. The expression $(n+1) \int_0^1 P_{n,k}(t) f(t)\, dt$ in the operators $M_n f$ takes the place of the expression $f(k/n)$ in $B_n f$, the Bernstein polynomials. These operators were introduced by Durrmeyer [6] and studied by Derriennic [2]. It was shown that $M_n f$ are positive contractions in L_p and are self-adjoint and commute, that is, $M_n M_k f = M_k M_n f$. These nice properties of $M_n f$ make them easier to work with. In this paper we will study the relation between derivatives of $M_n f$, the rate of approximation of M_n, and the smoothness of the function f. The smoothness of f is given, following [5], by

$$\omega_\varphi^r(f, t)_p = \sup_{0 < h \leqslant t} \|\Delta_{h\varphi}^r f\|_{L_p[0,1]}, \qquad \varphi(x) = (x(1-x))^{1/2}, \tag{1.2}$$

* Supported by NSERC of Canada, Grant A4816.

0021-9045/89 $3.00

where

$$\Delta_h^r f(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k f\left(x + \left(\frac{r}{2} - k\right) h\right), \quad \text{if } \left[x - \frac{rh}{2}, x + \frac{rh}{2}\right] \subset [0, 1],$$
$$\Delta_h^r f(x) = 0, \qquad \text{otherwise.} \tag{1.3}$$

We will construct an operator $O_n f$ using linear combinations of $M_n f$ and show, for $r > \alpha$ (and $\varphi(x)$ as above), that

$$\|O_n f - f\|_p = O(n^{-\alpha}) \Leftrightarrow \|\varphi^{2r} M_n^{(2r)} f\|_p = O(n^{r-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f, t)_p = O(t^{2\alpha}). \tag{1.4}$$

Section 2 will contain a short discussion of $\omega_\varphi^{2r}(f, t)_p$ and the related K-functionals. Section 3 will contain the necessary facts proved in earlier papers (see [2]) on $M_n f$. In Sections 4 and 5 we establish the estimates of $\|\varphi^{2r} M_n^{(2r)} f\|_p$ and $\|O_n f - f\|_p$ by $n^r \omega_\varphi^{2r}(f, 1/\sqrt{n})_p$ and $\omega_\varphi^{2r}(f, 1/\sqrt{n})_p$, respectively. The inverse results of (1.4) are obtained in Section 7, where

$$\|M_n f - f\|_p = O(n^{-\alpha}) \Leftrightarrow \omega_\varphi^2(f, t)_p = O(t^{2\alpha}), \qquad \alpha < 1, \tag{1.5}$$

is also established. We note that (1.5) is valid for $\alpha < 1$ while (1.4) is valid for $\alpha < r$.

2. Results on Moduli of Smoothness

For proof of our results we will utilize the K-functional characterization of $\omega_\varphi^r(f, t)_p$. The K-functional in question is given by

$$K_r(f, t^r)_p = \inf_g \{ \|f - g\|_p + t^r \|\varphi^r g^{(r)}\|_p \}, \tag{2.1}$$

where the infimum is taken on all g such that $g^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}$ (i.e., absolutely continuous in $[a, b]$ for every a, b satisfying $0 < a < b < 1$).

It was proved in [5, Chap. 2] that

$$M^{-1} \omega_\varphi^r(f, t)_p \leqslant K_r(f, t^r)_p \leqslant M \omega_\varphi^r(f, t)_p, \tag{2.2}$$

which we denote by $\omega_\varphi^r(f, t)_p \sim K_r(f, t^r)$. A different characterization of smoothness was given by $\tau_r(f, \psi(t))_{p,p}$,

$$\tau_r(f; \psi(t))_{p,p} = \left[\int_0^1 \frac{1}{\psi(t, x)} \int_{-\psi(t,x)}^{\psi(t,x)} |\Delta_v^r f(x)|^p \, dv \, dx\right]^{1/p}, \qquad p < \infty,$$
$$\tau_r(f; \psi(t))_\infty = \sup\{ |\Delta_h^r f(x)|; h \leqslant \psi(t, x), x \in [0, 1]\}, \tag{2.3}$$

where $\Delta_h^r f$ is given by (1.3) and $\psi(t, x) = t\varphi(x) + t^2$. It was shown in [8] that $\tau_r(f, \psi(t))_{p,p}$ is also equivalent to $K_r(f, t^r)_p$. The modulus $\tau_r(f, t)_p$ can be used in this paper as an alternative to $\omega_\varphi^r(f, t)_p$.

We define, for $\varphi(x) = (x(1-x))^{1/2}$,

$$\bar{K}_r(f, t^r)_p = \inf_{g^{(r-1)} \in \mathrm{AC}_{\mathrm{loc}}} (\|f-g\|_p + t^r \|\varphi^r g^{(r)}\|_p + t^{2r} \|g^{(r)}\|_p) \tag{2.4}$$

and we will also use the equivalence [5, Chap. 3]

$$\bar{K}_r(f, t^r)_p \sim K_r(f, t^r)_p \sim \omega_\varphi^r(f, t)_p \tag{2.5}$$

which is stronger than (2.2).

3. Properties of $M_n f$

For the convenience of the reader we will summarize here the properties of $M_n f$ and related formulae which will be needed later. Most of these can be found in [2].

A. $M_n f$ is a positive operator.

B. $M_n(1, x) = 1$, $M_n f$ preserves constants.

C. $\|M_n f\|_p \leqslant \|f\|_p$, $1 \leqslant p \leqslant \infty$, $M_n f$ is a contraction on $L_p[0, 1]$.

D. For $f \in L_1[0, 1]$

$$M_n(f, t) = \sum_{m=0}^{n} \lambda_{n,m} \left(\int_0^1 f(x)\, Q_m(x)\, dx \right) Q_m(t), \tag{3.1}$$

where Q_m is the Legendre polynomial of degree m, i.e.,

$$Q_m(x) = \frac{\sqrt{2m+1}}{m!} \left(\frac{d}{dx}\right)^m (x(1-x))^m \quad \text{for} \quad m \geqslant 1 \quad \text{and} \quad Q_0(x) = 1; \tag{3.2}$$

and

$$\lambda_{n,m} = \frac{(n+1)!\, n!}{(n-m)!\, (n+m+1)!}.$$

E. For a polynomial P_k of degree k $M_n P_k$ is a polynomial of degree $\min\{k, n\}$ [2, p. 337].

F. M_n commutes with M_k,

$$M_k M_n f = M_n M_k f = \sum_{m=0}^{\min(n,k)} \lambda_{n,m} \lambda_{k,m} \left(\int_0^1 f(x) Q_m(x)\, dx \right) Q_m. \tag{3.3}$$

G. For $f, g \in L_1[0, 1]$

$$\langle M_n f, g\rangle \equiv \int_0^1 M_n(f, t)\, g(t)\, dt = \int_0^1 f(t) M_n(g, t)\, dt \equiv \langle f, M_n g\rangle. \quad (3.4)$$

H. For $f \in L_1[0, 1]$

$$\left(\frac{d}{dx}\right)^r M_n(f, x) = (-1)^r \frac{(n+1)!\, n!}{(n-r)!\, (n+r)!} \sum_{k=0}^{n-r} P_{n-r,k}(x) \int_0^1 P^{(r)}_{n+r,k+r}(t) f(t)\, dt. \quad (3.5)$$

I. For $f \in L_1[0, 1]$, $f^{(2r-1)} \in \mathrm{AC}_{\mathrm{loc}}$, and $\varphi^{2r} f^{(2r)} \in L_1$ we have, using integration by parts,

$$\left(\frac{d}{dx}\right)^{2r} M_n(f, x) = \frac{(n+1)!\, n!}{(n-2r)!\, (n+2r)!} \sum_{k=0}^{n-2r} P_{n-2r,k}(x) \int_0^1 P_{n+2r,k+2r}(t) f^{(2r)}(t)\, dt, \quad (3.6)$$

which can be rewritten as

$$\varphi(x)^{2r} \left(\frac{d}{dx}\right)^{2r} M_n(f, x) = (n+1) \sum_{k=0}^{n-2r} \alpha(n, k) P_{n,k+r}(x) \int_0^1 P_{n,k+r}(t) \varphi(t)^{2r} f^{(2r)}(t)\, dt, \quad (3.7)$$

where

$$\alpha(n, k) = \frac{(k+r)!^2}{k!\, (k+2r)!} \frac{(n-k-r)!^2}{(n-k)!\, (n-k-2r)!} < 1.$$

4. The Direct Result for $\varphi^{2r} M^{(2r)} f$

To show the direct part of

$$\|\varphi^{2r} M_n^{(2r)} f\|_p = O(n^{r-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f, t)_p = O(t^{2\alpha}),$$

we will prove the following more general direct result.

THEOREM 4.1. *For* $f \in L_p[0,1]$, $1 \leqslant p \leqslant \infty$, $\omega_\varphi^{2r}(f,t)_p$ *given by* (1.2), *and* $\varphi(x)^2 = x(1-x)$ *we have*

$$\|\varphi^{2r} M_n^{(2r)} f\|_{L_p[0,1]} \leqslant M n^r \omega_\varphi^{2r}(f, 1/\sqrt{n})_p. \tag{4.1}$$

In fact, in view of (2.2), that is, $K_{2r}(f, t^{2r})_p \sim \omega_\varphi^{2r}(f,t)_p$, it will suffice to prove the two inequalities given in the following two lemmas:

LEMMA 4.2. *For* $g \in L_p[0,1]$, $g^{(2r-1)} \in \mathrm{AC}_{\mathrm{loc}}$, *and* $\varphi^{2r} g^{(2r)} \in L_p[0,1]$ *we have*

$$\|\varphi^{2r} M_n^{(2r)} g\|_{L_p[0,1]} \leqslant \|\varphi^{2r} g^{(2r)}\|_{L_p[0,1]}. \tag{4.2}$$

LEMMA 4.3. *For* $f \in L_p[0,1]$

$$\|\varphi^{2r} M_n^{(2r)} f\|_{L_p[0,1]} \leqslant M n^r \, \|f\|_{L_p[0,1]}. \tag{4.3}$$

Proof of Lemma 4.2. As a corollary of (3.7) and C of Section 3 we write

$$\begin{aligned}
\|\varphi^{2r} M_n^{(2r)} g\|_p &\leqslant \left\| (n+1) \sum_{k=0}^{n-2r} P_{n,k+r}(x) \int_0^1 P_{n,k+r}(t)\, \varphi(t)^{2r}\, g^{(2r)}(t)\, dt \right\|_p \\
&\leqslant \left\| (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_0^1 P_{n,k}(t)\, |\varphi(t)^{2r}\, g^{(2r)}(t)|\, dt \right\|_p \\
&= \|M_n(|\varphi^{2r} g^{(2r)}|)\|_p \leqslant \|\varphi^{2r} g^{(2r)}\|_p,
\end{aligned}$$

which completes the proof of the lemma.

Proof of Lemma 4.3. To prove (4.3) we divide $[0,1]$ into two parts, $E_n = [1/n, 1-1/n]$ and $E_n^c = [0, 1/n] \cup [1-1/n, 1]$, and prove (4.3) separately for $L_p(E_n)$ and $L_p(E_n^c)$. For the proof on $L_p(E_n^c)$ we observe

$$\left(\frac{d}{dt}\right)^{2r} M_n(f,t) = \frac{n!}{(n-2r)!} \sum_{k=0}^{n-2r} P_{n-2r,k}(t)\, \Delta^{2r} a_k(n), \tag{4.4}$$

where

$$a_k(n) = (n+1) \int_0^1 P_{n,k}(x) f(x)\, dx \tag{4.5}$$

and

$$\Delta^r a_k(n) = \Delta(\Delta^{r-1} a_k(n)) \qquad \text{and} \qquad \Delta a_k(n) = a_{k+1}(n) - a_k(n). \tag{4.6}$$

Since

$$\|\varphi(x)^{2r}\|_{L_\infty(E_n^c)} \leqslant n^{-r} \qquad \text{and} \qquad \frac{n!}{(n-2r)!} \leqslant n^{2r},$$

we have

$$\|\varphi^{2r}M_n^{(2r)}(f)\|_{L_p(E_n^c)} \leqslant \frac{n!}{(n-2r)!}\,n^{-r}\left\|\sum_{k=0}^{n-2r} P_{n-2r,k}(t)\,\Delta^{2r}a_k(n)\right\|_{L_p(E_n^c)}$$

$$\leqslant n^r \sum_{j=0}^{2r}\binom{2r}{j}\left\|\sum_{k=0}^{n-2r} P_{n-2r,k}(t)\,a_{k+j}(n)\right\|_{L_p[0,1]}$$

If we prove now for $0\leqslant j\leqslant 2r$

$$\left\|(n+1)\sum_{k=0}^{n-2r} P_{n-2r,k}(t)\int_0^1 P_{n,k+j}(x)\,f(x)\,dx\right\|_{L_p[0,1]} \leqslant C\,\|f\|_{L_p[0,1]}, \tag{4.7}$$

we will obtain

$$\|\varphi^{2r}M_n^{(2r)}(f)\|_{L_p(E_n^c)} \leqslant Cn^r 2^r\,\|f\|_{L_p[0,1]}.$$

To prove (4.7) for $p=\infty$, we observe that

$$\left\|(n+1)\sum_{k=0}^{n-2r} P_{n-2r,k}(t)\int_0^1 P_{n,k+j}(x)\,f(x)\,dx\right\|_{L_\infty[0,1]}$$

$$\leqslant \|f\|_\infty \sum_{k=0}^{n-2r} P_{n-2r,k}(x) = \|f\|_\infty.$$

For $p=1$ (recall $n\geqslant 2r\geqslant j$) we derive (4.7) by

$$\left\|(n+1)\sum_{k=0}^{n-2r} P_{n-2r,k}(t)\int_0^1 P_{n,k+j}(x)\,/f(x)|\,dx\right\|_{L_1[0,1]}$$

$$=\frac{n+1}{n-2r+1}\int_0^1\left\{\sum_{k=0}^{n-2r} P_{n,k+j}(x)\right\}|f(x)|\,dx \leqslant \left(\frac{n+1}{n-2r+1}\right)\|f\|_1.$$

Using now the Riesz–Thorin theorem, we establish (4.7) for $1\leqslant p\leqslant\infty$.

We now prove (4.3) in $L_p(E_n)$. It is sufficient to prove the result for $p=\infty$ and $p=1$ as we can use the Riesz–Thorin theorem to obtain from these special cases the result for $1<p<\infty$.

For L_∞ we follow the proof of Lemma 3.5 of [3, p. 282] observing that here we use $|(n+1)a_k(n)|\leqslant\|f\|_\infty$ (see (4.5)) instead of $|f(k/n)\leqslant\|f\|_\infty$.

For the L_1 estimate we again follow the proof of Lemma 3.5 of [3] and observe that we have to estimate terms of the type

$$I(l,m,n) = \|q_{l,m}(x)\,\varphi(x)^{2l-2r}n^l\sum_{k=0}^{n}(k-nx)^{2r-2l-m}a_k(n)\,P_{n,k}(x)\|_{L_1(E_n)},$$

where $q_{l,m}(x)$ is a polynomial in x independent of n or k and $0 \leqslant m \leqslant 2r-2l$, $0 \leqslant l \leqslant r$. We will show

$$\int_{E_n} \varphi(x)^{2l-2r} n^l \, |k-nx|^{2r-2l-m} \, P_{n,k}(x) \, dx \leqslant Cn^{r-1}, \tag{4.8}$$

and therefore

$$\begin{aligned} I(l, m, n) &\leqslant C_1 n^{r-1} \sum_{k=0}^{n} |a_k(n)| \\ &\leqslant C_1 n^r \sum_{k=0}^{n} \int_0^1 P_{n,k}(t) \, |f(t)| \, dt = C_1 n^r \, \|f\|_1. \end{aligned}$$

To prove (4.8) we recall that

$$|k-nx|^{2r-2l-m} \leqslant |k-nx|^{2r-2l} + 1$$

and

$$\int_{E_n} \varphi(x)^{2l-2r} n^l P_{n,k}(x) \, dx \leqslant n^r \int_0^1 P_{n,k}(x) \, dx \leqslant \frac{n^r}{n+1},$$

and therefore, it is sufficient to show

$$\int_{E_n} \varphi(x)^{-2s} (k-nx)^{2s} \, P_{n,k}(x) \, dx \leqslant C_2 n^{s-1}. \tag{4.9}$$

Inequality (4.9) was proved in [5, Chap. 9].

5. Direct Theorem for an Approximation Operator

In this section the direct theorem about an approximation operator will be proved pending several lemmas, which are of a technical nature and which will be proved in Section 6. The operators we utilize will satisfy

$$O_n f = O_n(f, x) = \sum_{i=0}^{2r-1} \alpha_i(n) M_{n_i}(f, x), \qquad n_0 = n < n_1 < \cdots < n_{2r-1} \leqslant An, \tag{5.1}$$

where A is independent of n,

$$O_n(1, x) = 1, \qquad O_n((\cdot - x)^m, x) = 0 \qquad \text{for} \quad m = 1, ..., 2r-1 \tag{5.2}$$

(which means that polynomials of order $2r-1$ are preserved), and

$$\sum_{i=0}^{2r-1} |\alpha_i(n)| \leqslant B, \tag{5.3}$$

where B is independent of n. Note that $O_n f$, $\alpha_i(n)$, A, and B change with choice of r. In the next section we will show in a constructive manner that such operators exist.

We will also need the following two lemmas which will be proved in the next section.

LEMMA 5.1. *For* $T_{n,m}(x) \equiv M_n((x-\cdot)^m, x)$ *we have*

$$|T_{n,2s}(x)| \leqslant Cn^{-s}\left(\varphi(x)^2+\frac{1}{n}\right)^s \qquad (\varphi(x)^2 = x(1-x)). \tag{5.4}$$

LEMMA 5.2. *For* $H_{n,m}(u)$ *given by*

$$\begin{aligned} H_{n,m}(u) \equiv (n+1)m\left\{\int_u^1 \int_0^u - \int_0^u \int_u^1\right\} \\ \times (u-t)^{m-1} \sum_{k=0}^{n} P_{n,k}(t)P_{n,k}(x)\,dt\,dx, \qquad m>0, \end{aligned} \tag{5.5}$$

we have

$$|H_{n,2s}(x)| \leqslant Cn^{-s}\left(\varphi(x)^2+\frac{1}{n}\right)^s \qquad (\varphi(x)^2 = x(1-x)). \tag{5.6}$$

We also need the following lemma.

LEMMA 5.3. *Suppose* $\Phi \in L_1, 0 \leqslant t \leqslant 1, \quad 0 \leqslant x \leqslant 1, \quad \alpha > 0,$ *and* $\varphi(x)^2 = x(1-x)$. *Then*

$$\left|\int_x^t (t-u)^{2r-1}\Phi(u)\,du\right| \leqslant \left|\frac{(t-x)^{2r-1}}{(\varphi(x)^2+\alpha)^r}\int_x^t (\varphi(u)^2+\alpha)^r\,|\Phi(u)|\,du\right|. \tag{5.7}$$

Proof. For $x=0$ or $x=1$ the result is trivial. For $0<x<u<t\leqslant 1$ we have $u>x$, $\alpha/(1-u)>\alpha/(1-x)$, and therefore, $1/(u+\alpha/(1-u))<1/(1+\alpha/(1-x))$. We also have $(t-u)/(1-u)<(t-x)/(1-x)$ and $t-u\leqslant t-x$, and therefore,

$$(t-u)^{2r-1} \leqslant \left(\frac{t-x}{1-x}\right)^r \left((t-u)\Big/\left(x+\frac{\alpha}{1-x}\right)\right)^{r-1} \frac{1}{(x+\alpha/(1-x))} (\varphi(u)^2+\alpha)^r$$
$$= \frac{(t-x)^{2r-1}}{(\varphi(x)^2+\alpha)^r} (\varphi(u)^2+\alpha)^r.$$

For $0 \leqslant t < u < x < 1$ we use $(u-t)/u < (x-t)/x$ and $1/(1-u+(\alpha/u)) < 1/(1-x+(\alpha/x))$ to obtain the same result.

We are now in a position to prove the direct result.

THEOREM 5.4. *Suppose $O_n f$ satisfies* (5.1), (5.2), *and* (5.3). *Then for* $1 \leqslant p \leqslant \infty$ *and* $\varphi(x)^2 = x(1-x)$

$$\|O_n f - f\|_p \leqslant L\omega_\varphi^{2r}(f, 1/\sqrt{n})_p. \tag{5.8}$$

Proof. We observe that (5.3) implies

$$\|O_n f - f\|_p \leqslant (B+1)\,\|f\|_p.$$

Using $\bar{K}_{2r}(f, t^{2r})_p \sim K_{2r}(f, t^{2r})_p$ proved in [5, Chap 3] (see also (2.1), (2.4), and (2.5)), it is now sufficient to show for $g^{(r-1)} \in \mathrm{AC}$ and $g^{(r)}$ in L_p that

$$\|O_n g - g\|_p \leqslant L_1 n^{-r} \left\| \left(\varphi^2 + \frac{1}{n}\right)^r g^{(2r)} \right\|_p$$
$$\leqslant L_2(n^{-r}\,\|\varphi^{2r} g^{(2r)}\|_p + n^{-2r}\,\|g^{(2r)}\|_p). \tag{5.9}$$

We expand g by the Taylor formula

$$g(t) = g(x) + (t-x)\,g'(x) + \cdots + \frac{(t-x)^{2r-1}}{(2r-1)!}\, g^{(2r-1)}(x) + R_{2r}(g, t, x),$$

where

$$R_{2r}(g, t, x) = \frac{1}{(2r-1)!} \int_x^t (t-u)^{2r-1}\, g^{(2r)}(u)\, du.$$

The identities in (5.2) now imply

$$O_n(g, x) - g(x) = O_n(R_{2r}(g, \cdot, x), x),$$

and therefore, using (5.1) and (5.3), it is enough to show

$$\|M_n(R_{2r}(g, \cdot, x), x)\|_p \leqslant L_3 n^{-r} \left\| \left(\varphi^2 + \frac{1}{n}\right)^r g^{(2r)} \right\|_p. \tag{5.10}$$

In fact we have to show this only for $p=\infty$ and $p=1$ and by the Riesz–Thorin theorem it will follow for $1<p<\infty$; but the proof for $p=\infty$ and that for $1<p\leqslant\infty$ are the same. Using the Riesz–Thorin theorem, however, it is clear that L_3 in (5.10) is independent of p. To prove (5.10) we recall the definition of the maximal function of ψ, $\mathscr{M}(\psi, x)$, given by

$$\mathscr{M}(\psi, x)=\sup_t \left|\frac{1}{t-x}\int_x^t |\psi(u)|\,du\right|.$$

We define for ψ given by $\psi(u)=(\varphi(u)^2+1/n)^r g^{(2r)}(u)$ for $u\in[0,1]$ and $\psi(u)=0$ otherwise, $\mathscr{M}(\psi, x)\equiv G(x)$. Using Lemma 5.3 with $\alpha=1/n$, we obtain

$$|M_n(R_{2r}(g,\cdot,x),x)|\leqslant\frac{1}{(2r-1)!}\left|M_n\left(\frac{(\cdot-x)^{2r}}{(\varphi(x)^2+1/n)^r}G(x),x\right)\right|,$$

and therefore,

$$\|M_n(R_{2r}(g,\cdot,x),x)\|_p\leqslant\|G\|_p\left\|M_n((\cdot-x)^{2r},x)\left(\varphi(x)^2+\frac{1}{n}\right)^{-r}\right\|_\infty.$$

We complete the proof for $1<p\leqslant\infty$ by recalling that the estimate

$$\left\|M_n((\cdot-x)^{2r},x)\left(\varphi(x)^2+\frac{1}{n}\right)^{-r}\right\|_\infty\leqslant Cn^{-r}$$

is in fact Lemma 5.1, and that the estimate

$$\|G\|_p\leqslant C_p\left\|\left(\varphi^2+\frac{1}{n}\right)^r g^{(2r)}\right\|_p$$

for $1<p\leqslant\infty$ is the well-known result on maximal functions. To prove (5.10) for $p=1$ we write, using Fubini's theorem,

$$\begin{aligned}&\int_0^1 |M_n(R_{2r}(g,\cdot,x),x)|\,dx\\&\quad\leqslant\frac{(n+1)}{(2r-1)!}\int_0^1\sum_{k=0}^n P_{n,k}(x)\int_0^1 P_{n,k}(t)\\&\qquad\times\left|\int_x^t (t-u)^{2r-1}\,|g^{(2r)}(u)|\,du\right|\,dt\,dx\end{aligned}$$

$$=\frac{n+1}{(2r-1)!}\int_0^1 |g^{(2r)}(u)| \left\{\int_u^1\int_0^u - \int_0^u\int_u^1\right\}$$

$$\times (u-t)^{2r-1} \sum_{k=0}^{n} P_{n,k}(t)P_{n,k}(x)\, dt\, dx\, du$$

$$=\frac{1}{(2r)!}\int_0^1 |g^{(2r)}(u)|\, |H_{n,2r}(u)|\, du.$$

Lemma 5.2 will now conclude the proof of (5.10) for $p=1$.

6. Lemmas on $O_n f$ and $M_n f$

As $O_n f$ would not be the operator with the minimum number of terms satisfying $\|O_n f - f\|_p \leqslant C\omega_\varphi^{2r}(f, 1/\sqrt{n})_p$ anyway, we will construct a relatively simple version of it. That is, we assume that $n_i = 2^i n$, $i=0, ..., 2r-1$. First we will need the following lemma.

Lemma 6.1. *For $M_n f$ given in* (1.1) *we have*

$$M_n((\cdot - x)^m; x) = \sum_{l=1}^{m} P_l(x) \prod_{i=1}^{l} (1/n+i+1), \qquad m=1, 2, ..., \tag{6.1}$$

where $P_l(x)$ are polynomials in x independent of n.

Proof. We calculate first $M_n(f_i, x)$ *for* $f_i(t)=t^i$,

$$M_n(f_i, x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x)\int_0^1 P_{n,k}(t)\, t^i\, dt$$

$$=(n+1)\sum_{k=0}^{n} P_{n,k}(x)\frac{n!(k+i)!}{(n+i)!k!}\int_0^1 P_{n+i,k+i}(t)\, dt$$

$$=\sum_{k=0}^{n} P_{n,k}(x)\frac{(k+1)\cdots(k+i)}{(n+2)\cdots(n+i+1)}.$$

We observe that

$$(k+1)\cdots(k+i) = k(k-1)\cdots(k-i+1) + \sum_{j=0}^{i-2} C_{j+1}\prod_{l=0}^{j}(k-l) + C_0,$$

where for $i=1$ the second term drops, and obtain

$$M_n(f_i, x) = ((n+2)\cdots(n+i+1))^{-1} \times \left[x^i n \cdots (n-i+1) + \sum_{j=0}^{i-2} C_{j+1} x^{j+1} n \cdots (n-j) + C_0 \right].$$

Expressing $n \cdots (n-l)$ in terms of $n+2$, $(n+2)(n+3)$, ... and their combinations and writing $(t-x)^m = \sum_{i=0}^{m} \binom{m}{i} (-x)^{m-i} f_i$, we obtain (6.1).

LEMMA 6.2. *For $n_i = 2^i n$, $i = 0, 1, ..., 2r-1$, there exist $\alpha_i(n)$ such that $O_n(f, x)$ given by* (5.1) *satisfies* (5.2) *and* (5.3) *and moreover $\alpha_i(n) \to \alpha_i$.*

Proof. Using Lemma 6.1, we have to calculate $\alpha_i(n)$ satisfying (for n big enough)

$$\sum_{i=0}^{2r-1} \alpha_i(n) = 1 \qquad \text{and} \qquad \sum_{i=0}^{2r-1} \alpha_i(n) \prod_{l=2}^{s} (2^i n + l)^{-1} = 0, \; s = 2, ..., 2r. \tag{6.2}$$

Using Cramer's rule and the Vandermonde determinant, we observe that for α_i satisfying

$$\sum_{i=0}^{2r-1} \alpha_i = 1 \qquad \text{and} \qquad \sum_{i=0}^{2r-1} \alpha_i 2^{-im} = 0, \qquad m = 1, ..., 2r-1, \tag{6.3}$$

we have a solution and that $\alpha_i(n) = \alpha_i + O(1/n)$.

To complete the proof of Theorem 5.4 we still have to extablish Lemma 5.1 and Lemma 5.2. We will prove a somewhat more general result which will be needed internaly in the proof of these lemmas as the proof is by induction. Lemma 5.1 is an immediate corollary of the following lemma.

LEMMA 6.3. *For $T_{n,m}(x) \equiv M_n((x-\cdot)^m, x)$ we have*

$$T_{n,2s}(x) = \sum_{i=0}^{s} p_{i,s,n}(x) \left(\frac{x(1-x)}{n} \right)^{s-i} n^{-2i} \tag{6.4}$$

and

$$T_{n,2s-1}(x) = \sum_{i=0}^{s-1} q_{i,s,n}(x) \left(\frac{x(1-x)}{n} \right)^{s-i-1} n^{-2i+1}, \tag{6.5}$$

where $q_{i,s,n}(x)$ and $p_{i,s,n}(x)$ are polynomials in x of fixed degree with coefficients that are bounded uniformly for all n.

Proof. The proof follows by a simple induction process from the known recursion relation [2],

$$(n+m+2)T_{n,m+1}(x)=x(1-x)[2mT_{n,m-1}(x)-T'_{n,m}(x)] \\ -(1-2x)(m+1)T_{n,m}(x) \tag{6.6}$$

and the fact that $T_{n,0}(x)=1$ and $T_{n,1}(x)=-(1-2x)/(n+2)$.

Remark. We can observe that $q_{i,s,n}(x)$ are divisible by $(1-2x)$ and that $p_{0,s,m}(x)$ and $q_{0,s,n}(x)/(1-2x)$ are constant in x.

Lemma 5.2 is an immediate corollary of the following lemma.

LEMMA 6.4. *Suppose $H_{n,m}(u)$ is given by (5.5) of Lemma 5.2. Then*

$$H_{n,m}(u)=n\sum_{k=1}^{n}P_{n+1,k}(u)\int_0^1(u-t)^mP_{n-1,k-1}(t)\,dt+u^{n+1}(u-1)^m \\ +(1-u)^{n+1}u^m, \tag{6.7}$$

$$(n+m+1)H_{n,m+1}(u)=u(1-u)[2mH_{n,m-1}(u)-H'_{n,m}(u)] \\ -(1-2u)mH_{n,m}(u), \tag{6.8}$$

$$H_{n,2s}(u)=\sum_{i=0}^{s-1}P_{i,s,n}(x)\left(\frac{x(1-x)}{n}\right)^{s-i}n^{-2i}, \qquad s>0, \tag{6.9}$$

and

$$H_{n,2s-1}(u)=\sum_{i=0}^{s-1}Q_{i,s,n}(x)\left(\frac{x(1-x)}{n}\right)^{s-i-1}n^{-2i+1}, \qquad s>0, \tag{6.10}$$

where $P_{i,n,s}(x)$ and $Q_{i,n,s}(x)$ are polynomials in x with coefficients bounded in n.

Proof. We first deduce (6.7) from (5.5), then (6.8) from (6.7), and after computing $H_{n,1}(u)$ and $H_{n,2}(u)$, (6.9) and (6.10) will follow (6.8) by induction, as (6.4) and (6.5) followed (6.6). We could have set $H_{n,0}(u)=1$ as indeed follows from (6.7), and as (6.8) is a corollary of (6.7), we may calculate only $H_{n,1}(u)$, but since $H_{n,0}(u)$ is not defined by (5.5), we calculate $H_{n,2}(u)$ as well.

We now write

$$H_{n,m}(u)=(n+1)m\left\{\int_u^1\int_0^u-\int_0^u\int_u^1\right\}(u-t)^{m-1}\sum_{k=0}^n P_{n,k}(x)P_{n,k}(t)\,dt\,dx$$

$$=m\int_0^u (u-t)^{m-1}\sum_{k=0}^n P_{n,k}(t)\,dt-(n+1)\,m\int_0^u\int_0^u (u-t)^{m-1}$$

$$\times\sum_{k=0}^n P_{n,k}(t)P_{n,k}(x)\,dt\,dx$$

$$-(n+1)m\int_0^u\int_0^1 (u-t)^{m-1}\sum_{k=0}^n P_{n,k}(t)P_{n,k}(x)\,dt\,dx$$

$$+(n+1)m\int_0^u\int_0^u (u-t)^{m-1}\sum_{k=0}^n P_{n,k}(t)P_{n,k}(x)\,dt\,dx$$

$$=u^m-(n+1)m\int_0^u\int_0^1 (u-t)^{m-1}\sum_{k=0}^n P_{n,k}(t)P_{n,k}(x)\,dt\,dx.$$

From the above we can easily calculate

$$H_{m,1}(u)=0 \qquad \text{and} \qquad H_{m,2}(u)=\frac{2}{n+2}u(1-u). \tag{6.11}$$

We now recall that $P'_{n,k}(x)=n(P_{n-1,k-1}(x)-P_{n-1,k}(x))$ and obtain, using integration by parts,

$$m\int_0^1 (u-t)^{m-1}P_{n,k}(t)\,dt$$

$$=-(u-1)^m P_{n,k}(1)+u^m P_{n,k}(0)+n\int_0^1 (u-t)^m$$

$$\times[P_{n-1,k-1}(t)-P_{n-1,k}(t)]\,dt.$$

Recalling that $P_{n,k}(1)=\delta_{n,k}$ and $P_{n,k}(0)=\delta_{0,k}$, and using the expression for $H_{n,m}(u)$ that we calculated, we have

$$H_{n,m}(u)=u^m+(u-1^m\,(n+1)\int_0^u x^n\,dx-u^m(n+1)\int_0^{u} (1-x)^n\,dx$$

$$-n(n+1)\int_0^u\int_0^1 (u-t)^m$$

$$\times\sum_{k=0}^n P_{n,k}(x)[P_{n-1,k-1}(t)-P_{n-1,k}(t)]\,dt\,dx$$

$$= (u-1)^m u^{n+1} + u^m (1-u)^{n+1}$$

$$- n(n+1) \int_0^1 (u-t)^m$$

$$\times \int_0^u \sum_{k=0}^{n-1} [P_{n,k+1}(x) - P_{n,k}(x)]\, P_{n-1,k}(t)\, dx\, dt$$

$$= (u-1)^m u^{n+1} + u^m (1-u)^{n+1}$$

$$+ n \int_0^1 (u-t)^m \sum_{k=0}^{n-1} P_{n+1,k+1}(u) P_{n-1,k}(t)\, dt,$$

and therefore, (6.7).

To prove (6.8) we write

$$L_{n,m}(u) \equiv u^{n+1}(u-1)^m + (1-u)^{n+1} u^m,$$

and observe that (6.8) is valid for $L_{n,m}(u)$ in place of $H_{n,m}(u)$. This is done using straightforward computations. That is,

$$u(1-u)\Big[2mu^{n+1}(u-1)^{m-1} + 2m(1-u)^{n+1}u^{m-1}$$

$$- \frac{d}{du}\{u^{n+1}(u-1)^m + (1-u)^{n+1}u^m\}\Big]$$

$$- (1-2u)m[u^{n+1}(u-1)^m + (1-u)^{n+1}u^m]$$

$$= m[-u^{n+2}(u-1)^m + (1-u)^{n+2}u^m]$$

$$+ (n+1)[u^{n+1}(u-1)^{m+1} + (1-u)^{n+1}u^{m+1}]$$

$$+ m[u^{n+1}(u-1)^{m+1} + u^{n+2}(u-1)^m$$

$$- (1-u)^{n+2}u^m + (1-u)^{n+1}u^{m+1}]$$

$$= (n+m+1)[u^{n+1}(u-1)^{m+1} + (1-u)^{n+1}u^{m+1}].$$

We now need to prove (6.8) for $H_{n,m}(u) - L_{n,m}(u) \equiv A_{n,m}(u)$. Recalling

$$u(1-u) P'_{n,k}(u) = (k-nu) P_{n,k}(u),$$

we write for $A_{n,m}(u)$ as above

$$u(1-u)\left[\frac{d}{du} A_{n,m}(u) - mA_{n,m-1}(u)\right]$$

$$= nu(1-u) \sum_{k=1}^{n} P'_{n+1,k}(u) \int_0^1 (u-t)^m P_{n-1,k-1}(t)\, dt$$

$$= n \sum_{k=1}^{n} P_{n+1,k}(u) \int_0^1 (u-t)^m (k-(n+1)u) P_{n-1,k-1}(t)\,dt$$

$$= n \sum_{k=1}^{n} P_{n+1,k}(u) \int_0^1 (u-t)^m \,[((k-1)-(n-1)t)$$

$$-(n-1)(u-t)+(1-2u)]\, P_{n-1,k-1}(t)\,dt$$

$$= n \sum_{k=1}^{n} P_{n+1,k}(u) \int_0^1 (u-t)^m t(1-t) P'_{n-1,k-1}(t)\,dt$$

$$-(n-1) A_{n,m+1}(u) + (1-2u) A_{n,m}(u)$$

$$= n \sum_{k=1}^{n} P_{n+1,k}(u) \int_0^1 \{m(u-t)^{m-1} t(1-t)$$

$$-(1-2t)(u-t)^m\} P_{n-1,k-1}(t)\,dt$$

$$-(n-1) A_{n,m+1}(u) + (1-2u) A_{n,m}(u)$$

$$= n \sum_{k=1}^{n} P_{n+1,k}(u) \int_0^1 \{-(m+2)(u-t)^2$$

$$-(m+1)(1-2u)(u-t) + mu(1-u)\}$$

$$\times (u-t)^{m-1} P_{n-1,k-1}(t)\,dt - (n-1) A_{n,m+1}(u)$$

$$+(1-2u) A_{n,m}(u) = -(n+m+1)\, A_{n,m+1}(u)$$

$$-m(1-2u) A_{n,m}(u) + mu(1-u) A_{n,m-1}(u),$$

which is what we wanted to prove.

7. The Inverse Results

In this section we will deduce the bahaviour of $\omega_\varphi^{2r}(f, t)_p$ from the behaviour of $\|\varphi^{2r} M_n^{(2r)} f\|_p$ or that of $\|\sum \alpha_i M_{n_i} f - f\|_p$.

Theorem 7.1. *Suppose* $f \in L_p[0, 1]$, $M_n f$ *is defined by* (1.1), $\varphi(x)^2 = x(1-x)$, *and* $\alpha \leqslant r$. *Then*

$$\|\varphi^{2r} M_n^{(2r)} f\| = O(n^{r-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f, t)_p = O(t^{2\alpha}). \tag{7.1}$$

Proof. The implication "$\Leftarrow$" was proved in Theorem 4.1, and in fact a more general result was shown. To prove the implication "$\Rightarrow$" we will show that for any k

$$\|\varphi^{2r} M_n^{(2r)} f\| \leqslant A n^{r-\alpha} \qquad \text{implies} \quad \omega_\varphi^{2r}(M_k f, t)_p \leqslant A \cdot B t^{2\alpha}, \tag{7.2}$$

where B may depend on r and α but not on k (and certainly not on n). This will be sufficient as we write

$$\omega_\varphi^{2r}(f, t)_p \leqslant \omega_\varphi^{2r}(f - M_k f, t)_p + \omega_\varphi^{2r}(M_k f, t)_p \leqslant C \|f - M_k\|_p + ABt^{2\alpha}$$

and let k tend to ∞.

Using [5], $\omega_\varphi^{2r}(f, t)_p \sim K_{2r}(f, t)_p$ (see (2.2)), and the definition of $K_{2r}(f, t^{2r})_p$, we have

$$K_{2r}(M_k f, t^{2r})_p \leqslant \|M_k f - O_n M_k f\|_p + t^{2r} \left\| \varphi(t)^{2r} \left(\frac{d}{dt}\right)^{2r} O_n(M_k f, t) \right\|_p.$$

We choose O_n to be that given in Theorem 5.4, and therefore,

$$\|M_k f - O_n M_k f\|_p \leqslant L_1 \omega_\varphi^{2r}(M_k f, 1/\sqrt{n})_p \leqslant L K_{2r}(M_k f, n^{-r})_p,$$

where L is independent of n and k. Moreover, using the definition of O_n, we have

$$\left\| \varphi(t)^{2r} \left(\frac{d}{dt}\right)^{2r} O_n(M_k f, t) \right\|_p \leqslant A_1 \max_{0 \leqslant i \leqslant 2r} \left\| \varphi(t)^{2r} \left(\frac{d}{dt}\right)^{2r} M_{2^i n}(M_k f, t) \right\|_p.$$

Lemma 4.2 and formula (3.3) will now imply

$$\begin{aligned} \left\| \varphi(t)^{2r} \left(\frac{d}{dt}\right)^{2r} M_l(M_k f, t) \right\|_p &= \left\| \varphi(t)^{2r} \left(\frac{d}{dt}\right)^{2r} M_k(M_l f, t) \right\|_p \\ &\leqslant \|\varphi^{2r} M_l^{(2r)} f\|_p \leqslant A l^{r-\alpha}. \end{aligned}$$

Writing $2^i n$ for l, and combining with the above, we obtain

$$K_{2r}(M_k f, t^{2r})_p \leqslant L_1 K_{2r}(M_k f, n^{-r})_p + A_2 t^{2r} n^{r-\alpha}.$$

We choose n such that $1/\sqrt{n} \leqslant t/R < 1/\sqrt{n-1}$ where R is to be chosen later and obtain, repeating the argument m times,

$$\begin{aligned} K_{2r}(M_k f, t^{2r})_p &\leqslant L_1 K_{2r}(M_k f, (t/R)^{2r})_p + A_2 R^{2r} t^{2\alpha} \\ &\leqslant L_1^2 K_{2r}(M_k f, (t/R)^{4r})_p + A_2 R^{2r} t^{2\alpha} + A_2 R^{2r} L_1 (t/R)^{2\alpha} \\ &\leqslant L_1^m K_{2r}(M_k f, (t/R)^{2mr})_p \\ &\quad + A_2 R^{2r} t^{2\alpha} (1 + L_1/R^{2\alpha} + \cdots + (L_1/R^{2\alpha})^{m-1}). \end{aligned}$$

We choose R ($R \geqslant 1$) such that $L_1/R^{2\alpha} \leqslant 1/2$ and recall, using Lemma 4.3, that

$$\begin{aligned} L_1^m K_{2r}(M_k f, (t/R)^{2mr})_p &\leqslant L_1^m (t/R)^{2mr} \|\varphi^{2r} M_k^{(2r)} f\|_p \\ &\leqslant L_1^m (t/R)^{2mr} C k^r \|f\|_p. \end{aligned}$$

Since k is fixed and $L_1/R^{2r} \leqslant 1/2$, we have

$$\lim_{m \to \infty} L_1^m K_{2r}(M_k f, (t/R)^{2mr})_p = 0.$$

Therefore,

$$K_{2r}(M_k f, t^{2r})_p \leqslant 2A_2 R^{2r} t^{2\alpha}$$

which concludes our proof.

THEOREM 7.2. *Suppose*

$$O_n f = \sum_{i=0}^{k_1} \alpha_i(n) M_{n_i} f, \quad n = n_0 < n_1 \cdots < n_k \leqslant An \qquad \text{and}$$
$$\sum |\alpha_i(n)| \leqslant B. \tag{7.3}$$

Then

$$\|O_n f - f\|_p = O(n^{-\alpha}) \qquad \text{implies} \quad \omega_\varphi^{2r}(f, t)_p = O(t^{2\alpha}) \qquad \text{for} \quad r > \alpha. \tag{7.4}$$

Remark. Note that the requirement that $O_n f$ actually approximates f is hidden in $\|O_n f - f\|_p = O(n^{-\alpha})$ and that condition (5.2) of Theorem 5.4 is not necessary.

Proof. Using the K functional (rather than $\omega_\varphi^{2r}(f, t)_p$), we write

$$K_{2r}(f, t^{2r})_p \leqslant \|O_n f - f\|_p + t^{2r} \|\varphi^{2r} O_n^{(2r)} f\|_p.$$

Theorem 4.1 now implies

$$\|\varphi^{2r} O_n^{(2r)} f\| \leqslant B \max_{n \leqslant m \leqslant An} \|\varphi^{2r} M_m^{(2r)} f\| \leqslant B \cdot M \omega_\varphi^{2r}(f, 1/\sqrt{n})_p$$
$$\leqslant C K_{2r}(f, n^{-r})_p.$$

As our assumption is $\|O_n f - f\|_p \leqslant An^{-\alpha}$, we have

$$K_{2r}(f, t^{2r})_p \leqslant An^{-\alpha} + Ct^{2r} K_{2r}(f, n^{-r})_p,$$

and this implies, via the Berens–Lorentz lemma [1], that if $\alpha < r$, $K_{2r}(f, t^{2r})_p \leqslant C_1 t^{2\alpha}$.

We now have as a corollary:

COROLLARY 7.3. *For $O_n f$ given in Theorem 5.4 and $\alpha < r$*

$$\|O_n f - f\|_p = O(n^{-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f, t)_p = O(t^{2\alpha}). \tag{7.5}$$

We also have as a result (partially a corollary) the following direct and inverse theorem on $M_n f$.

THEOREM 7.4. *For $M_n f$ given in* (1.1), $1 \leqslant p \leqslant \infty$, *and* $\alpha < 1$

$$\|M_n f - f\|_p = O(n^{-\alpha}) \Leftrightarrow \omega_\varphi^{2r}(f, t)_p = O(t^{2\alpha}) \tag{7.6}$$

and

$$\|M_n f - f\|_p \leqslant C(\omega_\varphi^2(f, 1/\sqrt{n})_p + n^{-1} \|f\|_p). \tag{7.7}$$

Proof. The implication "$\Rightarrow$" was shown in Theorem 7.2. Following the proof of Theorem 5.4, we see that for $f'' \in L_p[0, 1]$

$$\|M_n f - f\| \leqslant A \left[\frac{1}{n} \|f'\|_p + \frac{1}{n} \|\phi^2 f''\|_p + \frac{1}{n^2} \|f''\|_p \right].$$

For $1 \leqslant p < \infty$ $\|f'\|_p \leqslant B[\|f\|_p + \|\varphi^2 f''\|_p]$, as was shown in [5, Chap. 9] using the Hardy inequality. As $\|M_n f - f\|_p \leqslant 2 \|f\|_p$, the above implies for $1 \leqslant p < \infty$

$$\|M_n f - f\|_p \leqslant C_1(\bar{K}_2(f, t)_p + n^{-1} \|f\|) \leqslant C(K_2(f, t)_p + n^{-1} \|f\|),$$

which completes the proof for such p. For $p = \infty$ the moments fit exactly the conditions in [4], and therefore, the direct result is proved there.

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